# COMMENTS ON "FOUNDATIONS OF P-ADIC TEICHMÜLLER THEORY"

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(1.) In §0.6, §0.9 of the Introduction, the *captions* and *actual illustrations* of Fig. 3, 4 [i.e., the *complex* and *p-adic* cases] are *reversed.* The illustration in the $p$-adic case appears with the correct caption in Fig. 3 of §1.6 of [3].

(2.) In line 4 of Theorem 1.3, (3), of §1.2 of the Introduction, *"odd"* should read *"even"*. This result is stated correctly in the body of the text — e.g., Chapter IV, Theorem 3.2; Chapter V, §1.

(3.) In the statement of the criterion (*) in the discussion following Chapter I, Definition 2.8, the word *"horizontal"* in the second and third lines following the display should be replaced by *"vertical"*.

(4.) In Chapter II, §2.3, the *"versal families"* should be understood as being *possibly empty.* In particular, in item (1) of the statements of Chapter II, Theorem 2.8, Corollary 2.9, the phrase "smooth of dimension ..." should be replaced by "smooth of dimension ... (if it is not empty)" (cf. the phrasing of the final paragraph of the statement of Chapter II, Theorem 2.8).

(5.) In the second sentence of the third paragraph of §2.3.1 of the Introduction, the phrase "the hypercohomology of this complex" should read "the first hypercohomology module of this complex".

(6.) With regard to the notation "$\mathcal{N}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$", "$\mathcal{C}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$" in the paragraph immediately preceding Theorem 0.4 of §0.9 of the Introduction, we note the following: Let $K$ be a finite extension of $\mathbb{Q}_p$ and $\mathfrak{Y}$ a formally smooth $p$-adic formal scheme over the ring of integers $\mathcal{O}_K$ of $K$, i.e., such as a suitable étale localization of $\mathcal{N}$ or $\mathcal{C}$. Then $\mathfrak{Y} \times_{\mathbb{Z}_p} \mathbb{Q}_p$ (i.e., "$\mathfrak{Y} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$") is defined as the *ringed space* obtained by tensoring the structure sheaf of $\mathfrak{Y}$ over $\mathcal{O}_K$ with $K$. Thus, if, for instance, $\mathfrak{Y}$ is the formal scheme obtained as the formal inverse limit of an inverse system of schemes

$$\ldots \hookrightarrow \mathfrak{Y}_n \hookrightarrow \mathfrak{Y}_{n+1} \hookrightarrow \ldots$$

— where $n$ ranges over the positive integers, and each "$\hookrightarrow$" is a nilpotent thickening — and $U$ is an affine open of the *common* underlying topological space of the $\mathfrak{Y}_n$, then the rings of sections of the respective structure sheaves $\mathcal{O}_{\mathfrak{Y}}$, $\mathcal{O}_Y$ of $\mathfrak{Y}$, $Y$ over $U$ are, by definition, given as follows:

$$\mathcal{O}_{\mathfrak{Y}}(U) \stackrel{\text{def}}{=} \varprojlim_n \mathcal{O}_{\mathfrak{Y}_n}(U); \quad \mathcal{O}_Y(U) \stackrel{\text{def}}{=} \mathcal{O}_{\mathfrak{Y}}(U) \otimes_{\mathcal{O}_K} K.$$

Here, we observe that $\mathcal{O}_{\mathfrak{Y}}(U)$ is the *p-adic completion* of a *normal noetherian ring of finite type over* $\mathcal{O}_K$. In particular, we observe that one may consider *finite étale coverings* of $Y$, i.e., by considering *systems of finite étale algebras* $\mathcal{A}_U$ over the various $\mathcal{O}_Y(U)$ [that is to say, as $U$ is allowed to vary over the affine opens of the $\mathfrak{Y}_n$] equipped with *gluings* over the intersections of the various $U$ that appear. Note, moreover, that by considering the *normalizations* of the $\mathcal{O}_{\mathfrak{Y}}(U)$ in $\mathcal{A}_U$, we conclude [cf. the discussion of the Remark immediately following Theorem 2.6 in Section II of [1]] that

(NorFor) any such system $\{\mathcal{A}_U\}_U$ may be obtained as the $W \stackrel{\text{def}}{=} \mathfrak{W} \times_{\mathcal{O}_K} K$ for some *formal scheme* $\mathfrak{W}$ that is *finite* over $\mathcal{Y}$, and that arises as the *formal inverse limit* of an inverse system of schemes

$$\ldots \hookrightarrow \mathfrak{W}_n \hookrightarrow \mathfrak{W}_{n+1} \hookrightarrow \ldots$$

— where $n$ ranges over the positive integers; each "$\hookrightarrow$" is a nilpotent thickening; for each affine open $V$ of the *common* underlying topological space of the $\mathfrak{W}_n$, $\mathcal{O}_{\mathfrak{W}}(V)$ is the *p-adic completion* of a *normal noetherian ring of finite type over* $\mathcal{O}_K$.

Indeed, this follows from well-known considerations in commutative algebra, which we review as follows. Let $R$ be a *normal noetherian ring of finite type over a complete discrete valuation ring* $A$ [i.e., such as $\mathcal{O}_K$ in the above discussion] with *maximal ideal* $\mathfrak{m}_A$ and *quotient field* $F$ such that $R$ is *separated* in the $\mathfrak{m}_A$-adic topology. Thus, since $A$ is *excellent* [cf. [2], Scholie 7.8.3, (iii)], it follows [cf. [2], Scholie 7.8.3, (ii)] that $R$ is *excellent*, hence that the $\mathfrak{m}_A$-adic completion $\widehat{R}$ of $R$ is also *normal* [cf. [2], Scholie 7.8.3, (v)]. Then it is well-known and easily verified [by applying a well-known argument involving the *trace map*] that the *normalization* of $\widehat{R}$ in any *finite étale algebra* over $\widehat{R} \otimes_A F$ is a *finite algebra* over $\widehat{R}$. Let $\widehat{S}$ be such a *finite algebra* over $\widehat{R}$. Then it follows immediately from a suitable version of *"Hensel's Lemma"* [cf., e.g., the argument of [4], Lemma 2.1] that $\widehat{S}$ may be obtained, as the notation suggests, as the $\mathfrak{m}_A$-adic completion of a *finite algebra* $S$ over $R$, which may in fact be assumed to be *separated* in the $\mathfrak{m}_A$-adic topology and [by replacing $S$ by its normalization and applying [2], Scholie 7.8.3, (v), (vi)] *normal*. Let $f \in R$ be an element that maps to a *non-nilpotent* element of $R/\mathfrak{m}_A \cdot R$. Write $R_f \stackrel{\text{def}}{=} R[f^{-1}]$; $S_f \stackrel{\text{def}}{=} S \otimes_R R_f$; $\widehat{R}_f$, $\widehat{S}_f$ for the respective $\mathfrak{m}_A$-adic completions of $R_f$, $S_f$. Then it follows again from [2], Scholie 7.8.3, (v), that $\widehat{S}_f$, which may be naturally identified [since $S$ is a *finite algebra* over $R$] with $\widehat{S} \otimes_{\widehat{R}} \widehat{R}_f$, is *normal*. That is to say, it follows immediately that

(NorForZar) the operation of forming *normalizations* [i.e., as in the above discussion] is *compatible* with *Zariski localization* on the *given formal scheme*.

On the other hand, one verifies immediately that (NorFor) follows formally from (NorForZar).

## Bibliography

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